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Pressure fluctuations in a randomly permeable medium

I T Drummond and R R Horgan

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge University, Cambridge CB3 9EW, UK

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Abstract. We extend a diffusion simulation method to compute pressure fluctuation correlations for a fluid flowing in a randomly permeable medium.

1. Introduction

In a previous paper (Drummond and Horgan 1987) we studied the problem of computing the effective permeability for large-scale flows in a medium in which the local permeability fluctuates according to a known (or assumed) statistical distribution. Here we show that the same methods may be applied to the computation of correlations of fluctuations in the pressure about a constant average gradient. Pressure fluctuation correlations are of great practical interest since they reflect statistical properties of the structure of the medium. They may also provide information relevant to the thermodynamic history of the fluid flowing in the medium.

2. Diffusion method

The diffusion method which we used previously to compute the effective permeability of the random medium can be adapted to the computation of the pressure fluctuations. We will assume that a sample of the random medium is represented by a symmetric permeability tensor field $\kappa_{ij}(\boldsymbol{x})$. The velocity $u_i(\boldsymbol{x})$ of fluid flowing in the medium is related to the pressure distribution $P(\boldsymbol{x})$, by a local form of Darcy's law

$$u_i(\boldsymbol{x}) = -\kappa_{ij}(\boldsymbol{x})\partial_j P(\boldsymbol{x}). \tag{2.1}$$

The flow is incompressible so the pressure satisfies the equation

$$\partial_i \kappa_{ij}(\boldsymbol{x})\partial_j P(\boldsymbol{x}) = 0. \tag{2.2}$$

In circumstances where there is a constant average pressure gradient $\boldsymbol{g}$ the pressure distribution is given by

$$P(\boldsymbol{x}) = \boldsymbol{g}\cdot\boldsymbol{x} + p(\boldsymbol{x}) \tag{2.3}$$

where $p(\boldsymbol{x})$, on average, is zero. From (2.2) we find that

$$\partial_i \kappa_{ij}(\boldsymbol{x})\partial_j p(\boldsymbol{x}) = -\partial_i \kappa_{ij}(\boldsymbol{x}) g_j. \tag{2.4}$$

The Green function for the medium, which vanishes for large values of its arguments, satisfies

$$\partial_j \kappa_{ij}(x) G(x, x') = -\delta(x - x'). \tag{2.5}$$

It then follows from (2.4) that

$$p(x) = R_j(x) g_j \tag{2.6}$$

where

$$R_j(x) = \int d^3x' G(x, x') \partial'_i \kappa_{ij}(x'). \tag{2.7}$$

The vector field $R_j(x)$, therefore, allows us to compute the pressure fluctuations by simply obtaining its projection along the direction of the pressure gradient.

As pointed out in our previous paper (Drummond and Horgan 1987), the above flow problem can be associated with a diffusion problem specified by the equation

$$\partial_i \kappa_{ij}(x) \partial_j P(x, \tau) = \frac{\partial}{\partial \tau} P(x, \tau) \tag{2.8}$$

where now we can think of $P(x, \tau)$ as a probability density. (Of course, τ is an artificial time variable.) The connection between the two problems is made precise by introducing $F(x, x', \tau)$ as that solution of (2.8) which satisfies the boundary condition

$$F(x, x', 0) = \delta(x - x'). \tag{2.9}$$

The Green function for the flow problem can be expressed as

$$G(x, x') = \int_0^\infty d\tau F(x, x'\tau). \tag{2.10}$$

Note that because of the symmetry of the permeability tensor both $G(x, x')$ and $F(x, x', \tau)$ are symmetric in their spatial arguments.

The connection between the two problems can be exploited to interpret $R_j(x)$ in terms of the diffusion problem. Consider a small (point-like) cloud of particles released at position x at $\tau = 0$, which then diffuses according to (2.10). Denote by $R_j(x, \tau)$ the displacement of the centre of mass of the cloud at time τ. We will show that

$$R_j(x) = \lim_{\tau \to \infty} R_j(x, \tau). \tag{2.11}$$

From its definition

$$R_j(x, \tau) = \int d^3x' (x' - x)_j F(x', x, \tau). \tag{2.12}$$

Using the relation

$$R_j(x, \tau) = \int_0^\tau d\tau' \frac{\partial}{\partial \tau'} R_j(x, \tau') \tag{2.13}$$

together with the fact that $F(x', x, \tau)$ satisfies (2.8) (with $x \to x'$) we find that

$$R_j(x, \tau) = \int_0^\tau d\tau' \int d^3x' (x' - x)_j \partial'_k \kappa_{kl}(x') \partial'_l F(x', x, \tau'). \tag{2.14}$$

Integrating by parts, we obtain the results

$$R_j(x, \tau) = \int_0^\tau d\tau' \int d^3x' F(x', x, \tau')\partial_i' \kappa_{ij}(x'). \tag{2.15}$$

Now using (2.10) we see that (2.11) follows immediately.

The pressure fluctuation correlation function is defined as

$$H(x - x') = \langle p(x)p(x') \rangle \tag{2.16}$$

where the angular brackets indicate an average taken over the ensemble representing the medium. We find, using the isotropy of the medium, that

$$H(x - x') = \tfrac{1}{3}g^2\langle \boldsymbol{R}(x) \cdot \boldsymbol{R}(x') \rangle. \tag{2.17}$$

The diffusion method for computing $H(x - x')$ then amounts to releasing clouds of particles at various points in the medium, measuring the vector field $\boldsymbol{R}(x)$ at these points, forming the appropriate scalar products and then averaging over different realisations of the medium.

3. Model random medium

We test the idea of the previous section on a model random medium of a type considered previously (Drummond and Horgan 1987). For simplicity we assume local isotropy for the medium, i.e.

$$\kappa_{ij}(x) = \delta_{ij}\kappa(x). \tag{3.1}$$

We further assume (log-normal statistics) that $\kappa(x)$ has the form

$$\kappa(x) = \kappa_0 \exp(\lambda\phi(x)) \tag{3.2}$$

where $\phi(x)$ is a Gaussian random field with zero mean and unit variance. Prescribing the two-point correlation function for $\phi(x)$ completes the model. We define

$$\Delta(x - x') = \langle \phi(x)\phi(x') \rangle \tag{3.3}$$

so

$$\Delta(0) = 1. \tag{3.4}$$

The parameter to κ_0 sets the scale for the permeability while λ controls the magnitude of relative fluctuations. The mean permeability is

$$\kappa_m = \kappa_0 \exp(\tfrac{1}{2}\lambda^2). \tag{3.5}$$

The assumption of homogeneity and isotropy for the statistical properties of the medium implies that the correlation function Δ depends only on $|x - x'|$.

Our method of constructing $\phi(x)$ for the purposes of simulation is the same as previously explained (Drummond and Horgan 1987). We set

$$\phi(x) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos(\boldsymbol{k}_n \cdot x + \varepsilon_n) \tag{3.6}$$

where the $\{\varepsilon_n\}$ and the $\{\boldsymbol{k}_n\}$ are independent random variables; ε_n is a uniformly distributed random phase and each $\boldsymbol{k}_n$ is selected from a probability distribution $D(\boldsymbol{k})$

which is essentially the former transform of $\Delta(\boldsymbol{x})$. For simplicity of simulation (and to make a point), we restrict ourselves in this paper to the case

$$D(\boldsymbol{k}) \propto \delta(k - k_0). \tag{3.7}$$

This leads to the result

$$\Delta(\boldsymbol{x}) = \sin k_0(|\boldsymbol{x}|)/k_0(|\boldsymbol{x}|) \tag{3.8}$$

so k_0^{-1} is a correlation length for the permeability fluctuations in the medium. The number of modes, N, does not affect the form of $\Delta(\boldsymbol{x})$. It need only be sufficiently large to ensure that the statistics of $\phi(\boldsymbol{x})$ is reasonably Gaussian. For our present purposes of demonstrating our method we do not have to be too strict about this. We found that $N = 8$ was adequate and computationally convenient.

4. Perturbation-theory calculation

In a given sample of the medium, the results for $R_j(\boldsymbol{x})$ and $R_j(\boldsymbol{x}, \tau)$ can be computed as a perturbation series in λ, i.e. in the fluctuations of the permeability. Even the lowest-order results are interesting since they give some feeling for the relationship between the fluctuation correlations $H(\boldsymbol{x} - \boldsymbol{x}')$ and the structure of the medium as represented by $\Delta(\boldsymbol{x} - \boldsymbol{x}')$.

The lowest approximation to $F(\boldsymbol{x}, \boldsymbol{x}', \tau)$ is

$$F_0(\boldsymbol{x}, \boldsymbol{x}', \tau) = \frac{1}{(4\pi\kappa_0\tau)^{3/2}} \exp\left(-\frac{(\boldsymbol{x} - \boldsymbol{x}')^2}{4\kappa_0\tau}\right) \tag{4.1}$$

i.e.

$$F_0(\boldsymbol{x}, \boldsymbol{x}', \tau) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \exp[\mathrm{i}\boldsymbol{q} \cdot (\boldsymbol{x} - \boldsymbol{x}') - \kappa_0\tau q^3]. \tag{4.2}$$

From equation (2.16) we may infer that, in our model, the lowest approximation to $R_j(\boldsymbol{x}, \tau)$ is

$$R_j(\boldsymbol{x}, \tau) = \lambda\kappa_0 \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}^3\boldsymbol{x}' F_0(\boldsymbol{x}, \boldsymbol{x}', \tau')\partial'_j\phi(\boldsymbol{x}'). \tag{4.3}$$

Using (4.2) we find

$$R_j(\boldsymbol{x}, \tau) = \lambda \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{x}) \left(\frac{1 - \exp(-\kappa_0 q^2\tau)}{q^2}\right) \mathrm{i}q_j\tilde{\phi}(\boldsymbol{q}) \tag{4.4}$$

where $\tilde{\phi}(\boldsymbol{q})$ is the Fourier transform of $\phi(\boldsymbol{x})$. Taking the limit $\tau \to \infty$, we see that

$$R_j(\boldsymbol{x}) = \lambda \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{x}) \left(\frac{\mathrm{i}q_j}{q^2}\right) \tilde{\phi}(\boldsymbol{q}). \tag{4.5}$$

Note that this result is independent of the overall scale κ_0, of the permeability.

When $\phi(\boldsymbol{x})$ is given by (3.5), we find

$$R_j(\boldsymbol{x}, \tau) = -\lambda \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \left(\frac{1 - \exp(\kappa_0 k_n^2\tau)}{k_n^2}\right) (\boldsymbol{k}_n)_j \sin(\boldsymbol{k}_n \cdot \boldsymbol{x} + \varepsilon_n). \tag{4.6}$$

For our particular model in which $|\boldsymbol{k}_n| = k_0$ for all n, we can easily see from (4.4)

$$R_j(\boldsymbol{x}, \tau) = [1 - \exp(-\kappa_0 k_0^2\tau)]R_j(\boldsymbol{x}). \tag{4.7}$$

That is, in our simple single-scale model, in lowest-order perturbation theory, $R_j(x, \tau)$ deviates from its asymptotic value by an exponentially decreasing amount. We can expect $R_j(x, \tau)$ to be a good approximation to its asymptotic value when

$$\kappa_0 k_0^2 \tau \geqslant 6.0. \tag{4.8}$$

This result has the following interpretation. The dispersion of the diffusing cloud is, in our approximation, $6\kappa_0\tau$. Equation (4.8) implies then that $R_j(x, \tau)$ is a good approximation to $R_j(x)$ when the size of the cloud is roughly six correlation lengths in size. This is intuitively plausible and is likely to hold true beyond lowest-order perturbation theory. It implies that the longer the length scale of the structures in the medium, the longer it is necessary to diffuse in order to detect them.

The corresponding low-order approximation for the pressure fluctuation correlation function is

$$H(\boldsymbol{x}-\boldsymbol{x}') = \tfrac{1}{3} g^2 \lambda^2 \kappa_0 \int \mathrm{d}^3 \boldsymbol{x}'' G_0(\boldsymbol{x}-\boldsymbol{x}'') \Delta(\boldsymbol{x}''-\boldsymbol{x}'). \tag{4.9}$$

Where

$$G_0(\boldsymbol{x}-\boldsymbol{x}'') = \frac{1}{4\pi\kappa_0 |\boldsymbol{x}-\boldsymbol{x}''|}. \tag{4.10}$$

For the single-scale model presently under consideration, this is easily evaluated to yield

$$H(\boldsymbol{x}-\boldsymbol{x}') = \tfrac{1}{3} g^2 \lambda^2 \left(\frac{1}{k_0^2}\right) \frac{\sin k_0 |\boldsymbol{x}-\boldsymbol{x}'|}{k_0 |\boldsymbol{x}-\boldsymbol{x}'|}. \tag{4.11}$$

Actually the single-scale model is a little misleading since it rigorously excludes wavevectors $\boldsymbol{k}_n \simeq 0$. In general these will be present. Their strength is measured by

$$\bar{\Delta}(\boldsymbol{k}) = \int \mathrm{d}^3 \boldsymbol{x} \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \Delta(\boldsymbol{x}) \tag{4.12}$$

for $k = 0$. They give rise to a long-range component in the pressure fluctuation correlations. From (4.9) we see that

$$H(\boldsymbol{x}-\boldsymbol{x}') \sim \frac{g^2 \lambda^2}{12\pi} \frac{\bar{\Delta}(0)}{|\boldsymbol{x}-\boldsymbol{x}'} \tag{4.13}$$

for large values of $|\boldsymbol{x}-\boldsymbol{x}'|$.

Again a long-range effect of this kind is presumably true beyond low-order perturbation theory and may be important in understanding some of the effects of pressure fluctuations.

5. Diffusion simulation

In our model the diffusion equation (2.8) simplifies to

$$\frac{\partial P}{\partial \tau} = \nabla \cdot (\kappa(\boldsymbol{x}) \nabla P). \tag{5.1}$$

The probability distribution $P(x, \tau)$ which satisfies this equation describes the statistical properties of a cloud of particles each of which moves according to the stochastic differential equation (in discrete approximation)

$$\Delta x = \nabla\kappa(x)\Delta\tau + \sqrt{2\kappa(x)\Delta\tau}\,\eta \tag{5.2}$$

where $\Delta\tau$ is the small discrete time step and the components of η are independent random variables with zero mean and unit variance.

We study the diffusion of particles in two stages. First we investigate the mean displacement of a cloud of particles, diffusing according to (5.2), and released from a particular point in a specific example of the random medium. A typical component of $R(x, \tau)$ is shown in figure 1, plotted as a function of τ. For $\lambda \leqslant 0.5$ the results of the simulation, shown as dots, are clearly consistent with the predictions of low-order perturbation theory obtained by evaluating (4.6) for the particular sample of the medium, which are shown as curves. For $\lambda > 0.5$, lowest-order perturbation theory is no longer accurate but the general nature of the results does not change dramatically. In figure 2 we plot the corresponding component of $R(x)$, against a range of values of λ. Again we see that predictions of low-order perturbation theory are very accurate for $\lambda \leqslant 0.5$. To obtain the statistical accuracy of these results it was necessary to follow the paths of 48 000 particles.

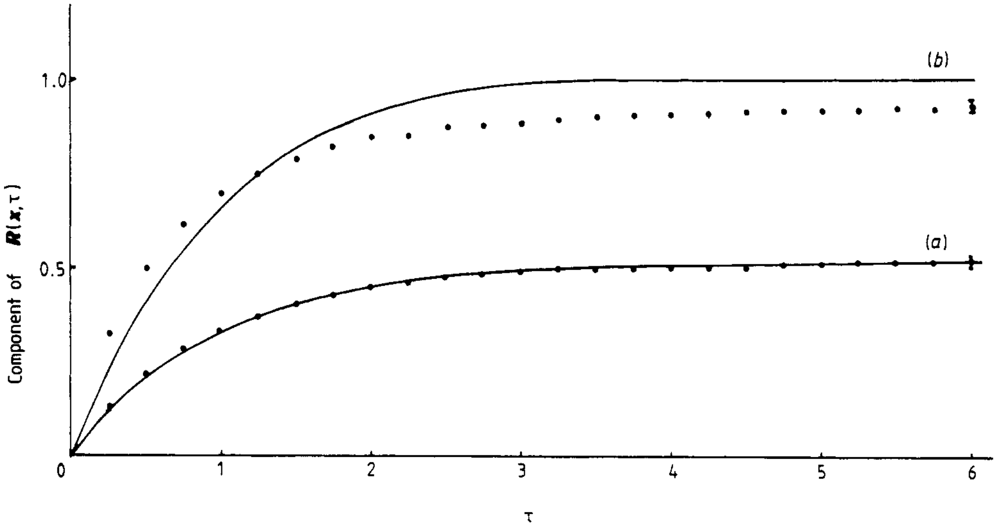

Figure 1. A component of the mean displacement of a cloud of particles released from a point in a particular medium shown as a function of τ. The predictions of first-order perturbation theory are shown as curves. The results of the simulation are shown as dots with error bars indicated on the last dot. Case (a) $\lambda = 0.5$; case (b) $\lambda = 1.0$.

In the second part of the investigation we released clouds of particles at various separations within the medium, measured $R(x)$ at each position x, formed the scalar products $R(x) \cdot R(x')$ for various pairs of points x and x', as required by (2.17), and finally formed the average over a number of samples of the medium.

The results of the simulation for $H(x - x')$ are compared with the prediction of lowest-order perturbation theory (4.11) in figure 3. Clearly simulation and theory are in excellent agreement. The statistical accuracy of these results required us to use

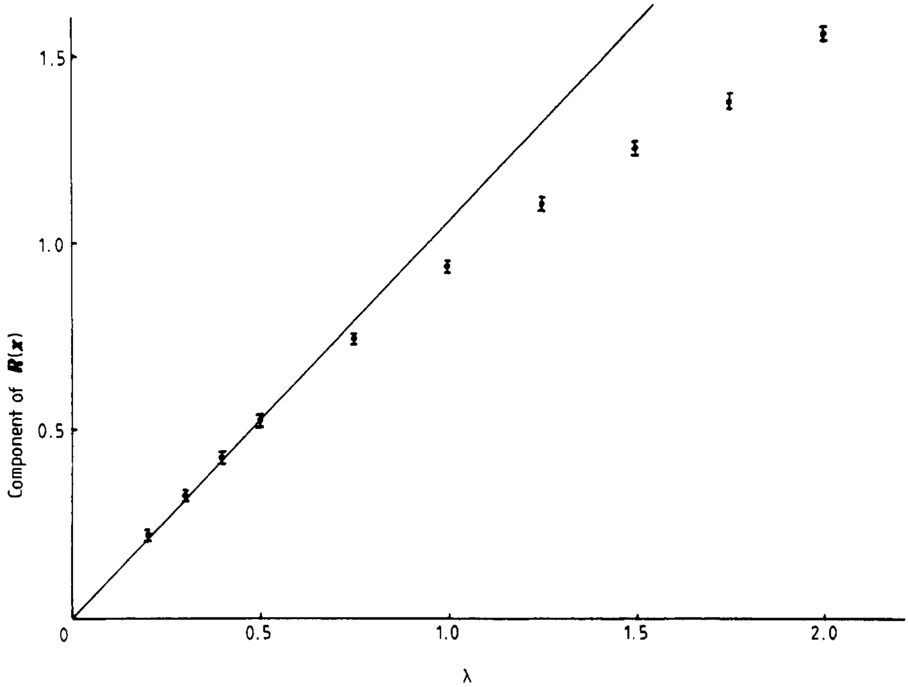

Figure 2. The limiting value of a component of the mean displacement of a cloud of particles in a particular medium as a function of λ. The straight line is the prediction of first-order perturbation theory.

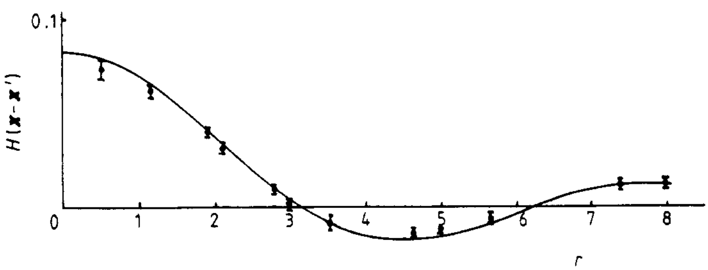

Figure 3. The pressure fluctuation correlation function for $\lambda = 0.5$ as a function of the separation. The curve is the prediction of first-order perturbation theory and the dots with error bars are the results of the simulation.

clouds of roughly 400 particles and 200 samples of the medium. There is a trade-off between the averaging over the particles in a cloud and averaging over samples of the random medium. The calculations were performed on a Convex C1 computer.

6. Conclusions

We have shown how the diffusion method may be used for simulating the pressure fluctuations present in steady flow through a random medium. The simulation worked well in our single-scale model and reproduced the predictions of perturbation theory for a reasonable range of permeability fluctuations. This gives us good grounds for believing that the simulation method can be used to evaluate other more complicated models and to explore situations where perturbation theory is not applicable.

An obvious direction for further investigation is that of multi-scale models, in particular those with long-range structure in the permeability distributions. A particularly interesting example of this is one where the permeability correlation function has an inverse power-law behaviour at large separations

$$\Delta(\boldsymbol{x}) \sim \frac{1}{|\boldsymbol{x}|^n} \tag{6.1}$$

for some constant n. Such behaviour may well lead to a breakdown of Darcy's law for global flow. In the diffusion approach this would show as anomalous diffusion.

Other models which merit detailed study are those in which isotropy breaks down at the local or global level. We believe this method can be usefully applied in these circumstances.

It is also possible to use the method to investigate the velocity field of the flow as well as the pressure field. This is important for studying the dispersion of material carried in the flow. To obtain information on the velocity field it is necessary to compute not only the displacement field $R_j(\boldsymbol{x})$ which determines the pressure fluctuations but also its derivatives $R_{j,k}(\boldsymbol{x})$. An accurate simulation can be achieved by regarding $R_{j,k}$ as a strain matrix and computing it in terms of a triad of unit vectors carried by each particle in the diffusion process. This is similar to calculations already performed by the authors for turbulent diffusion of magnetic fields (Drummond and Horgan 1986).

We feel then that our simulation approach, given sufficient computing power, should prove to be a useful way of analysing the properties of flow in random materials.

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